



Keyboards and Tuning

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Topics

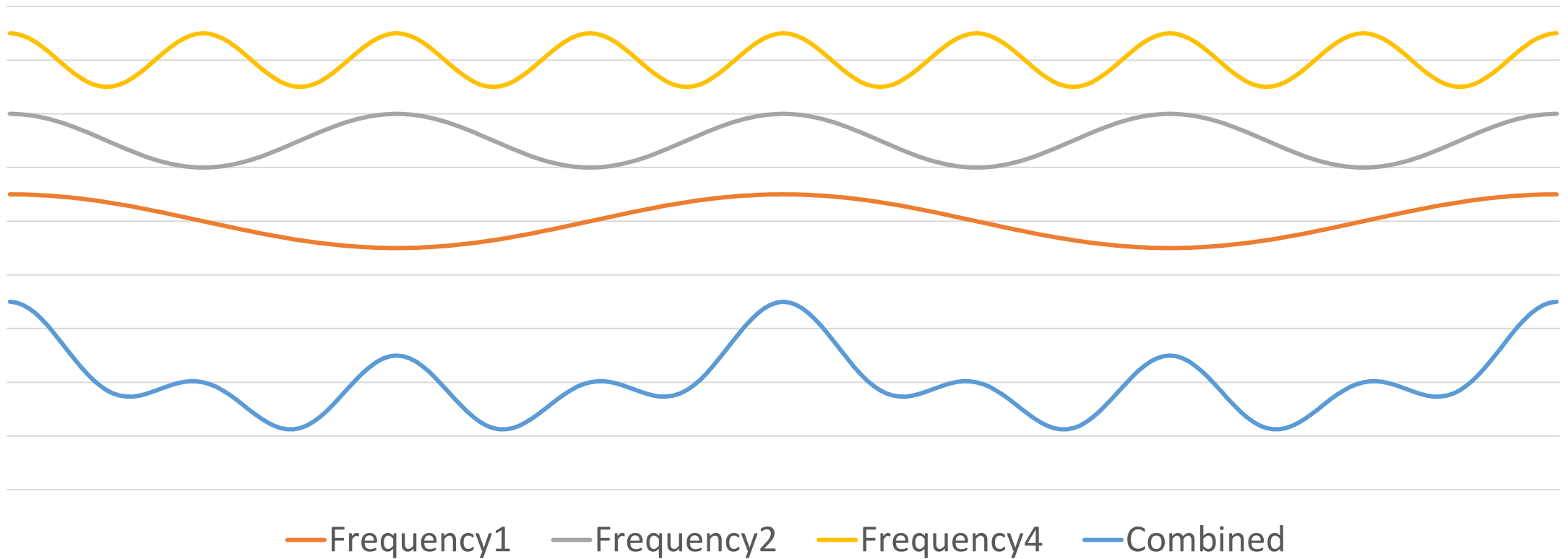
- Theory of harmony (sound wave frequencies and ratios)
- Key spacing options for the keyboard
 - Repeating patterns for front and back row keys
 - Even vs. uneven spacing of keys
 - Reason that the current keyboard has survived 650 years
- Tuning of the keys
 - Improving harmony with linear instead of $\log_2$ tuning
 - Setting tones using $(1 + n/w)$ instead of $2^{(n/w)}$
 - Improving tone spacing using jointed linear segments
 - Half-linear: 6 steps of $1/15$ followed by 6 steps of $1/10$
- Applying flexible tuning to electronic keyboards

Theory of harmony

- A single tone has a very simple repeating pattern like a sine wave
- Multiple tones combine into more complex patterns
- Tones with very complex ratios have long repeat lengths
- Ears can easily recognize short, simple, repeating cycles
- Tones 1, 2, 4, 8, etc. are doubles (*octaves*) and combine simply
 - Repeat length is short and equals the highest individual tone
- Next slide shows wavelengths 4, 2, 1 and how those combine (bottom)
- Y axis shows the air pressure waves that enter your ears

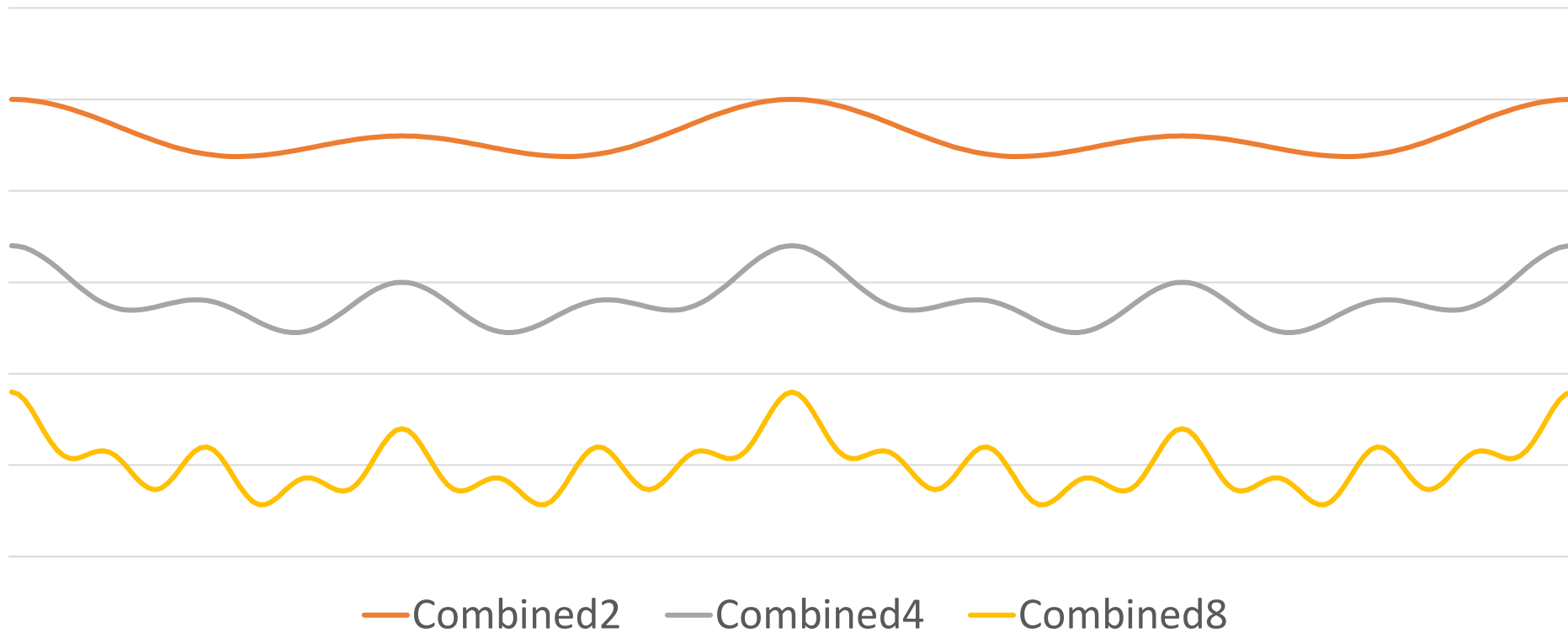
Harmony from combining 3 doubles (*octaves*)

Sound Frequencies 4, 2, 1, and Combined

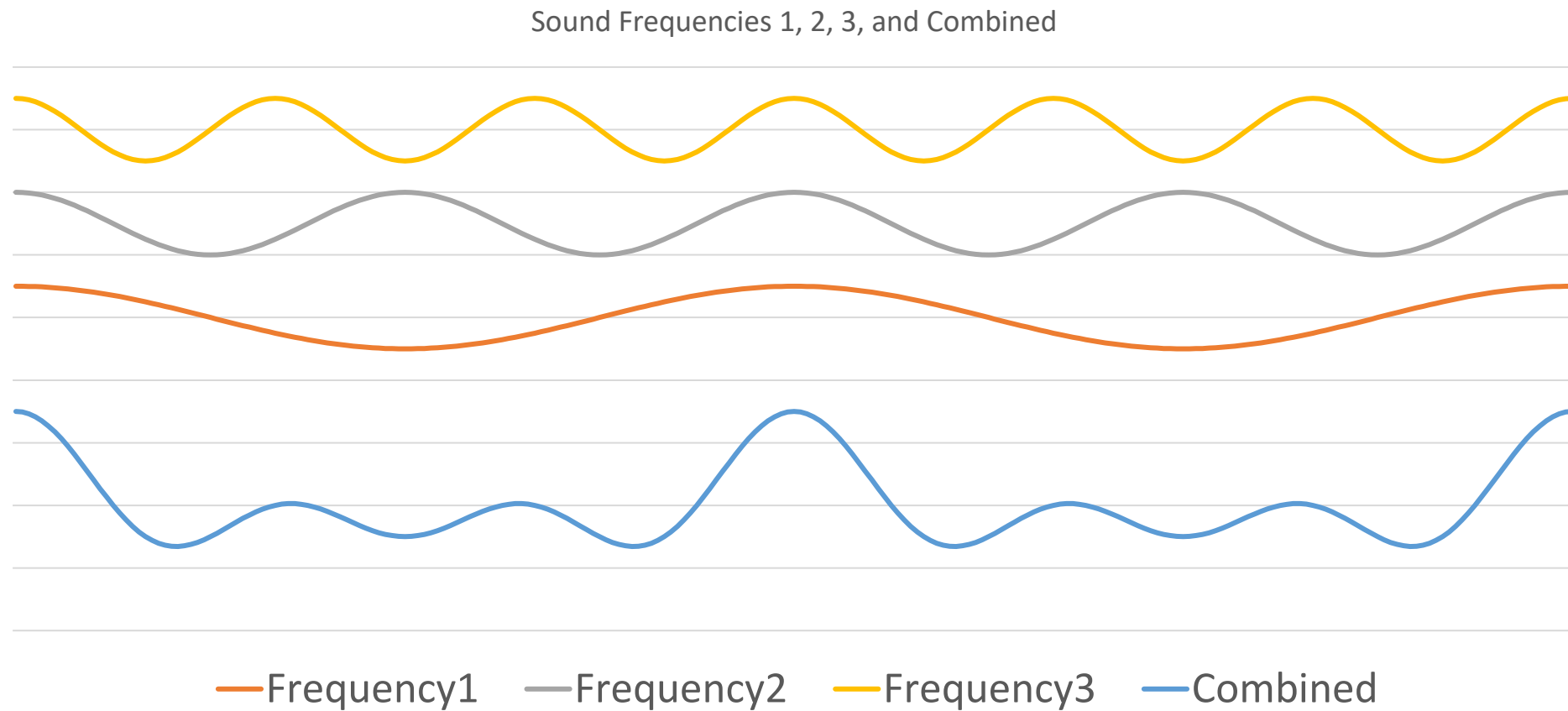


Combining 2, 3, or 4 doubles

Combining (1,2), (1,2,4), or (1,2,4,8)



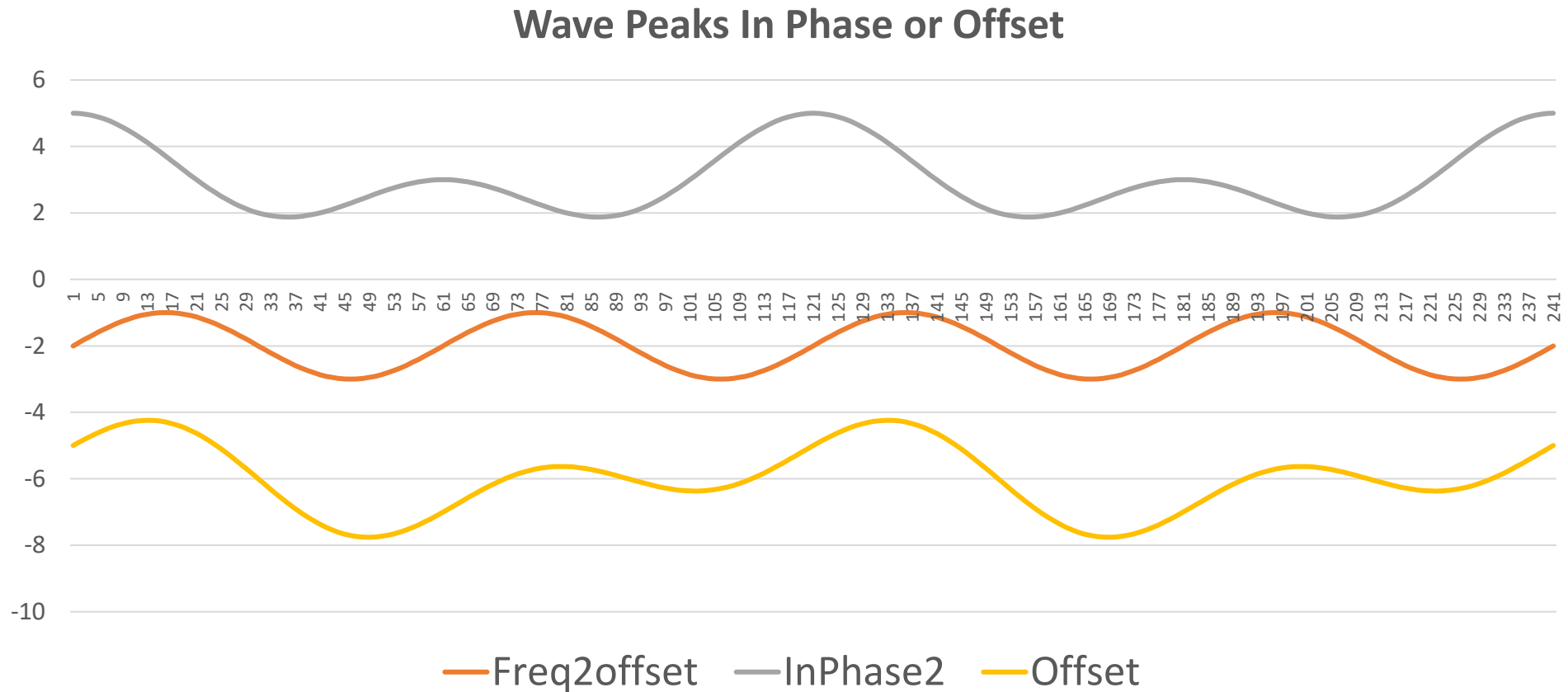
Harmony from combining frequencies 1, 2, and 3



Wave peaks that are synchronized or offset

- Even when exactly in tune, the timing of 2 pressure waves can differ.
- The peaks for 2 tones can be exactly synchronized or offset.
- The shape of waves is affected by when the 2 patterns reach your ears.
- Sound travels at about 1100 feet/second, more in warm or less in cold air.
- Distance between waves is 2.5 feet for standard A with 440 waves/second.
- Perfect harmony can match both frequency and timing of pressure peaks.
- Synchronized or offset wave combinations are in next slide (top, bottom).

Tone waves synchronized or offset (ratio 2:1)

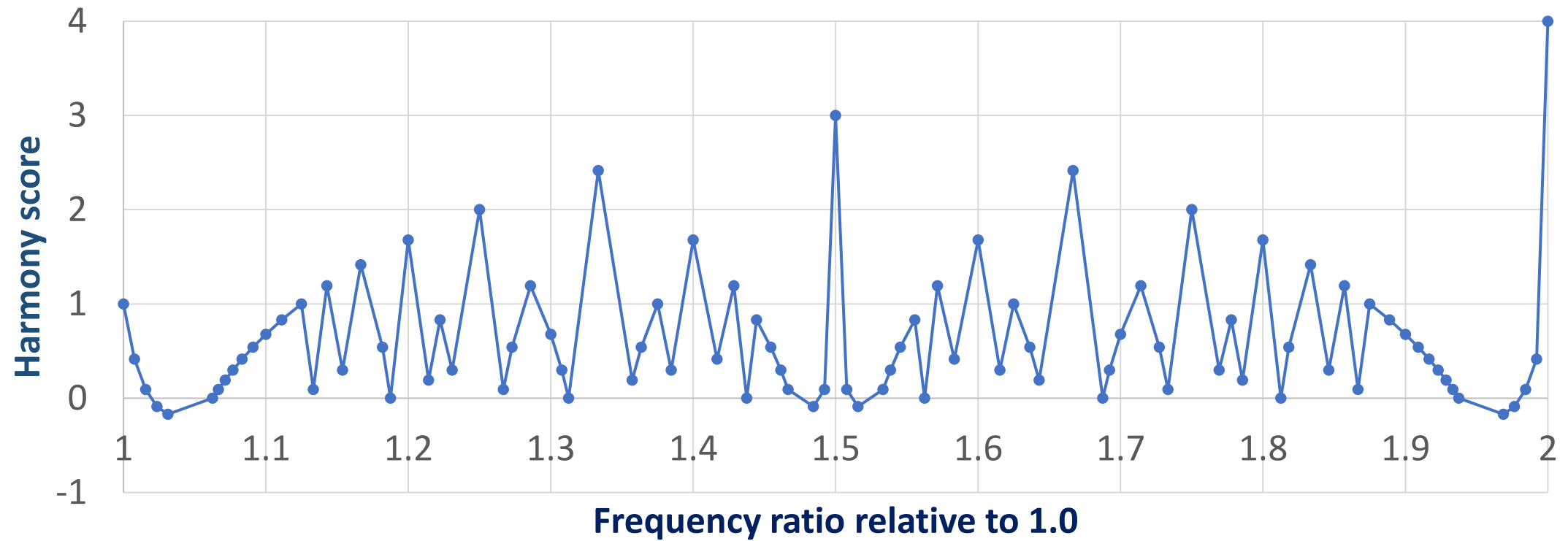


Which ratios sound best or worst?

- All frequency ratio choices between 1 and 2 can be rated and graphed.
- Rating: Divide the shorter of the two wavelengths by the wave repeat length.
 - For example, tone frequencies 2 and 3 have a repeat length of 6.
 - Frequency 3 has a shorter wavelength than 2.
 - Dividing wavelength 3 by repeat length 6 gives ratio $1:2 = 0.5$
- Then take log base 2 of that ratio and add 4.
 - For example, $-1 + 4 = 3$ is the rating for combining wavelengths 2 and 3.
 - Tone ratio 2 with 3 is equivalent to playing note C with the G above it.
 - Other tone ratios with longer repeat lengths get poorer ratings.
- See rating graph on next slide.

Harmony of 2 tones with ratios from 1 to 2

Best Harmony Choices Between 1 and 2



Observations about harmony graph

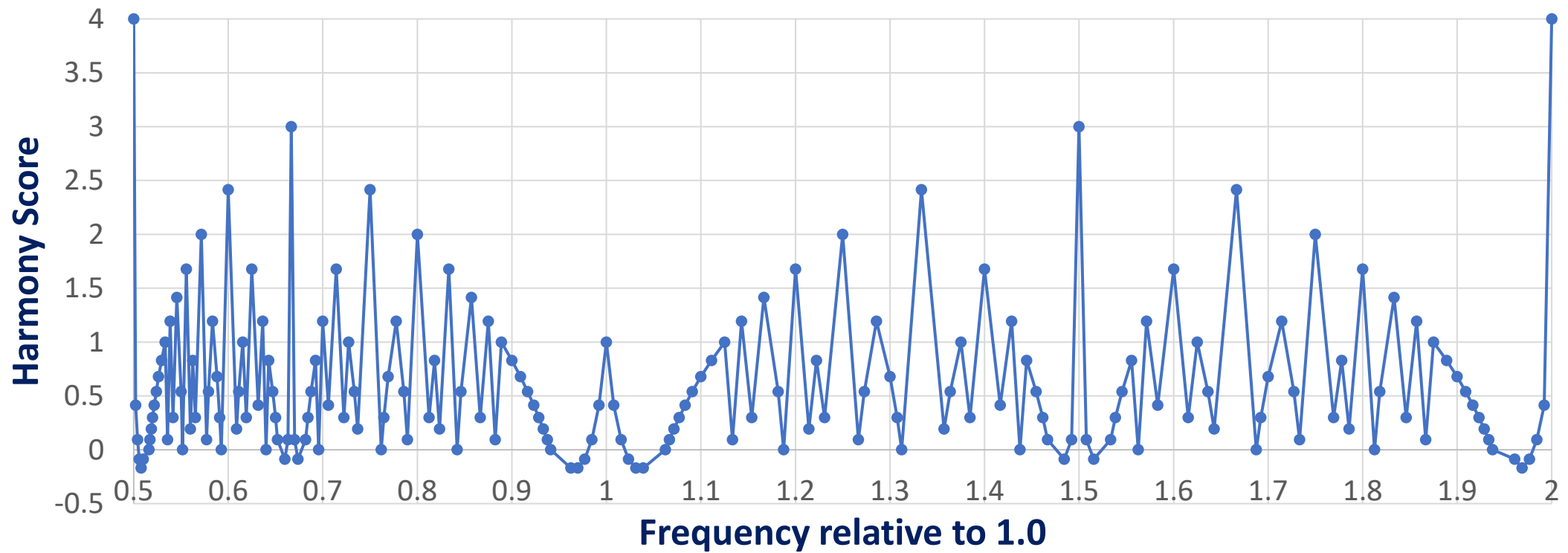
- Best harmony ratios are $2/1$, then $3/2$, then $4/3$, $5/3$, then $5/4$, $7/4$
- Poorest harmony choices are near the best choices
- Frequency ratio 1.03 is the poorest choice and gets the poorest rating
- Ratios such as 1.01 closer to 1.0 sound almost like a single tone
- Only about 10 precise ratios sound better than a single tone
- Most random combinations of 2 tones sound worse than a single tone
- A solo sounds better than a duet with poor harmony
- Precise harmony sounds better than a solo

Reciprocal harmony ratios

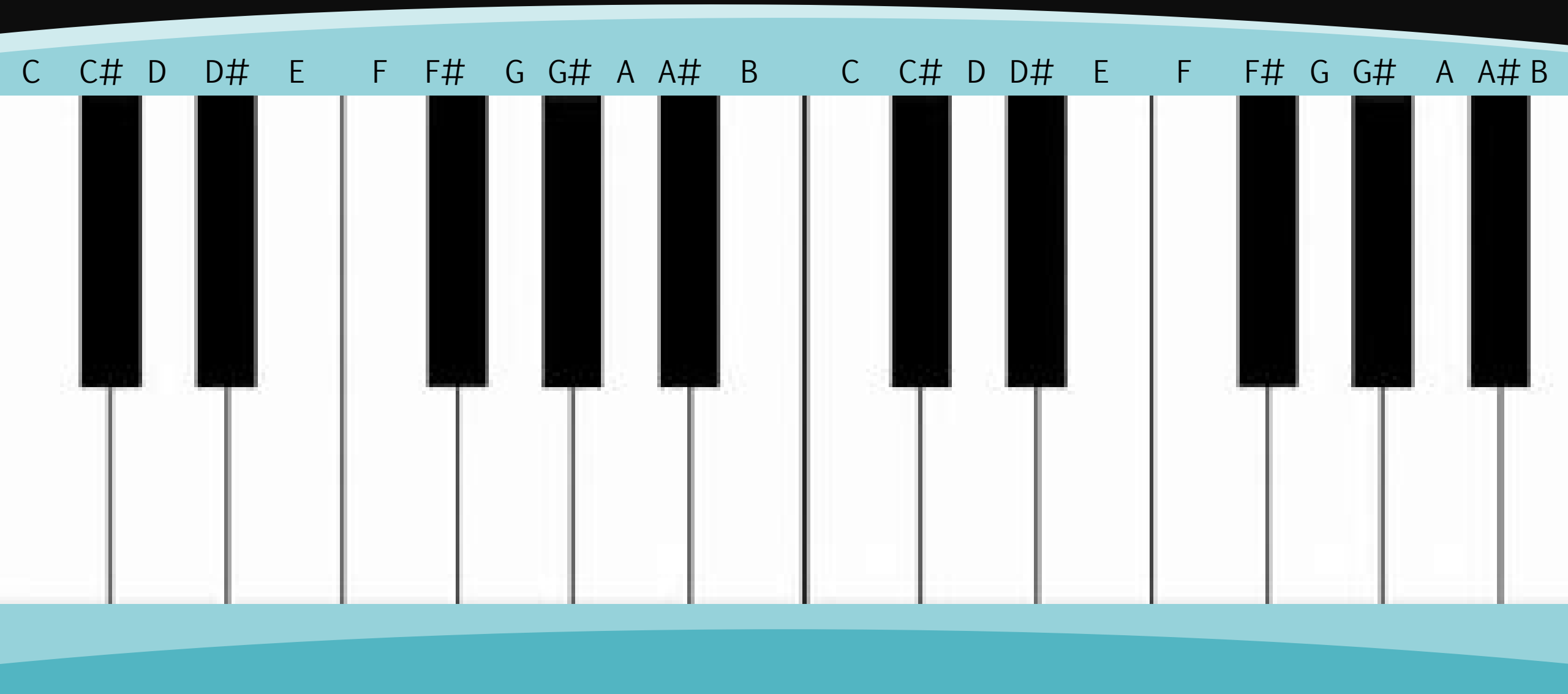
- The previous graph showed harmony choices above a standard tone.
- The best harmony choice between 1 and 2 was 1.5 (harmony ratio 3:2).
- But you may want to choose a lower instead of higher tone.
- The best harmony choice between 0.5 and 1 is 0.66667 (harmony ratio 2:3)
- If the standard tone is C, the G above it has ratio 1.5 with ratio 3:2.
- The best tone below C is the next F with ratio 2:3, not the G!
- Reciprocal tone ratios are not symmetrical above or below the standard.
- The next graph includes harmony choices below the standard tone.

Harmony of 2 tones with ratios from 0.5 to 2

Best Harmony Choices from 0.5 to 2



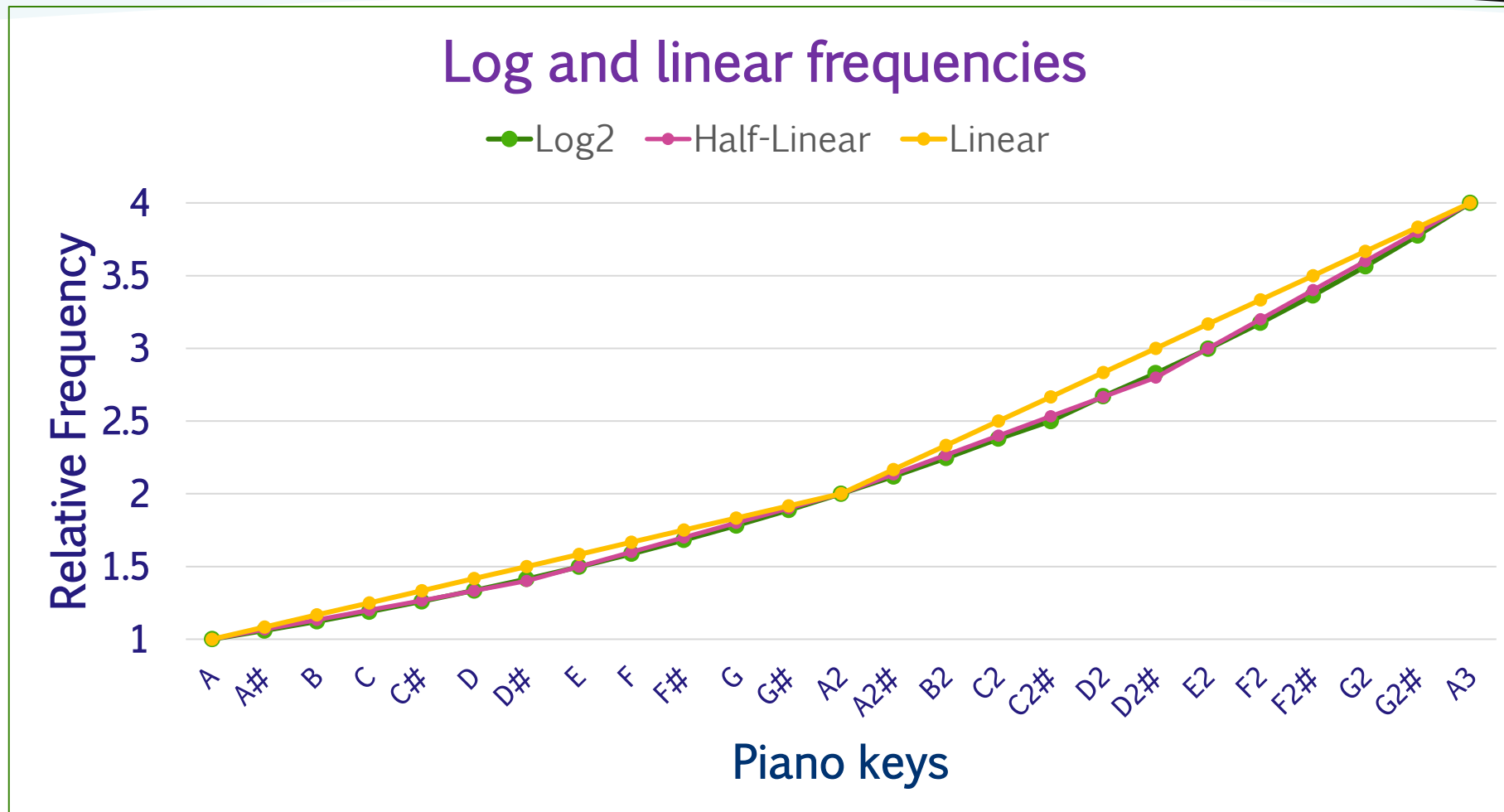
The standard keyboard from year 1350:
Unevenly spaced keys with 7 in front + 5 in back = 12



Linear vs. $\log_2$ tuning with 12 tones per double
 Half-linear uses 6 steps of $1/15$ and 6 of $1/10$

	Note Letter	C	D	E	F	G	A	B	C
	Note Number	0	2	4	5	7	9	11	12
Freq	$\log_2$	1	1.122	1.260	1.335	1.498	1.682	1.888	2
Freq	Half-linear	1	1.125	1.25	1.333	1.5	1.667	1.875	2
Freq	Linear	1	1.167	1.333	1.416	1.583	1.75	1.917	2
Ratio	Half-linear	1	$9/8$	$5/4$	$4/3$	$3/2$	$5/3$	$15/8$	2
Ratio	Linear	1	$7/6$	$4/3$	$17/12$	$19/12$	$7/4$	$23/12$	2

Frequencies across 2 doubles using lines within each double, half-double, or with $\log_2$ tuning



Reciprocal, linear, $\log_2$ and tuning frequencies across doubles ($w = 4$ notes per double)

Tuning	n	-4	-3	-2	-1	0	1	2	3	4
	Formula:					Base Note				
Linear	$1 + n/w$	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
Log2	$2^{(n/w)}$	0.5	0.595	0.707	0.841	1	1.189	1.414	1.682	2
Recip.	$2w/(2w-n)$	0.667	0.727	0.8	0.889	1	1.143	1.333	1.6	2
Linear	Ratio	0	1/4	1/2	3/4	1	5/4	3/2	7/4	2
Log2	Ratio	1/2	*	$\sqrt{(1/2)}$	*	1	*	$\sqrt{2}$	*	2
Recip.	Ratio	2/3	8/11	4/5	8/9	1	8/7	4/3	8/5	2

* Ratios for log tuning include the n/w root of 2 (the $\frac{1}{4}$ root or $\frac{3}{4}$ root)

Advantages of half-linear tuning, using 2 linear segments per doubling of frequency

- Includes the most pleasing 4-part harmony with ratios of 4 : 5 : 6 : 7 : 8
- Adjacent notes have common denominators to simplify frequency ratios
- The 2 denominators also have simple ratios
- Numerators in both segments have ratios of 15 : 16 : 17 : 18 : 19 : 20
- The base note was shifted to C by multiplying all ratios by 15/16.
- The 2 linear segments join at E which can be represented as 15/12, 20/16, or 5/4 and at A# which can be represented as 21/12, 28/16, or 7/4.
- Music with simple ratios sounds better than with standard tuning.

Tone frequencies with **half-linear** tuning, 12 tones per double, showing cents (1/100 of 2^{-12})

Key	0	1	2	3	4	5	6	7	8	9	10	11	12
Letter	C	C#	D	D#	E	F	F#	G	G#	A	A#	B	C
Ratios	1	17/16	9/8	19/16	5/4	4/3	17/12	3/2	19/12	5/3	7/4	15/8	2
Series	16/16	17/16	18/16	19/16	20/16	16/12	17/12	18/12	19/12	20/12	21/12	30/16	32/16
Joints	15/12					28/16							
Freq	1	1.063	1.125	1.188	1.25	1.333	1.417	1.5	1.583	1.667	1.75	1.875	2
Cents	0	105	204	298	386	471	603	702	796	884	969	1088	1200
Piano	0	100	200	300	400	500	600	700	800	900	1000	1100	1200
Diff	0	+5	+4	-2	-14	-29	+3	+2	-4	-16	-31	-12	0

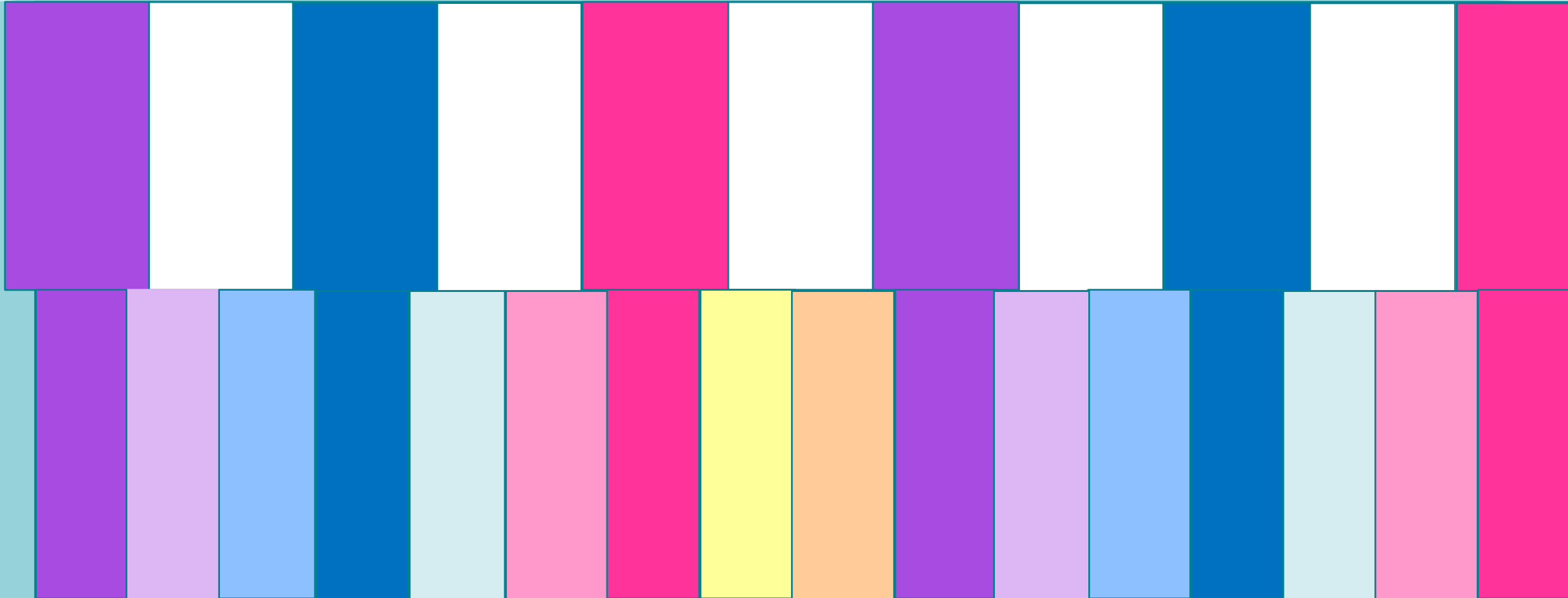
Cents are used for tuning piano keys.

Joints are where the 6-note straight lines intersect.

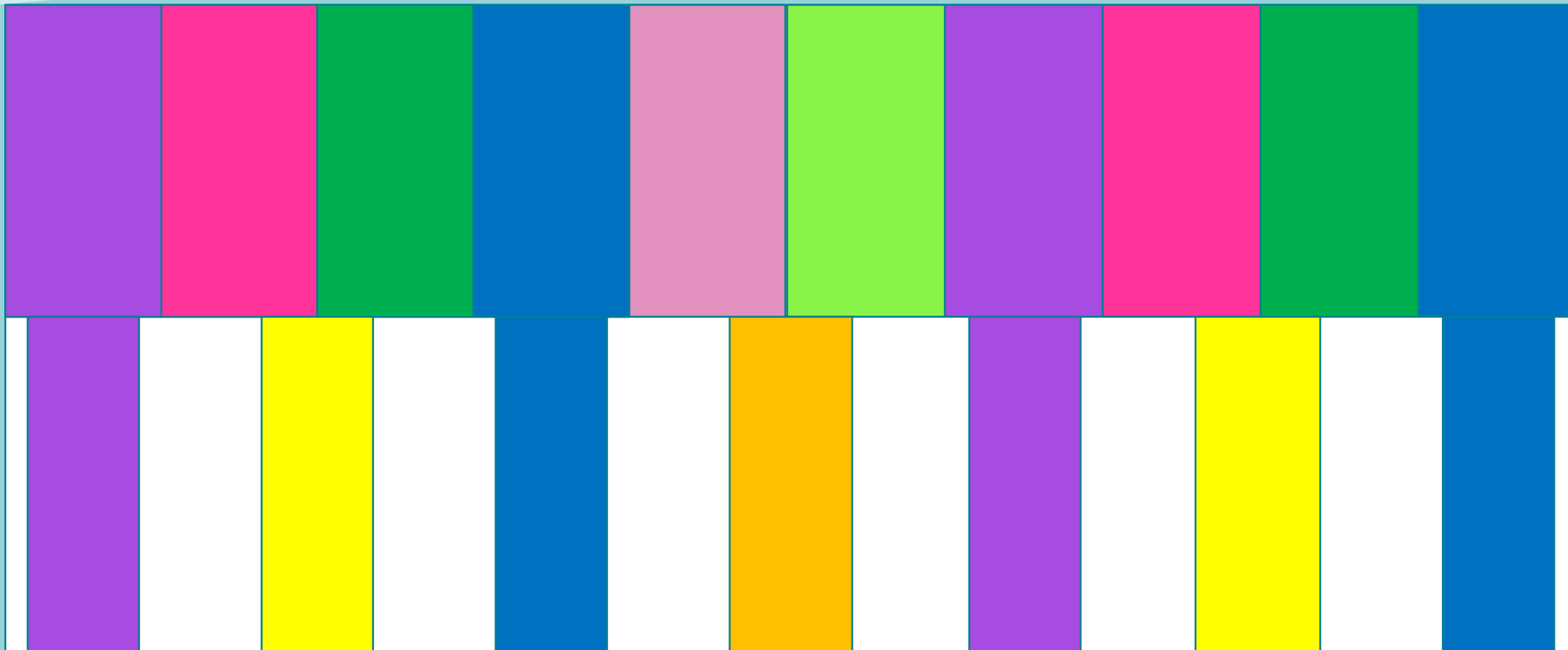
Evenly spaced keys are much better for math

- Electronic keyboards can allow many tuning options.
- Standard keyboards also have useful features:
 - The same key pattern within each double.
 - Two rows of keys let fingers reach more notes.
 - Texture of keys help fingers find the notes.
- New keyboards can keep those features and add new options:
 - Options for 12 keys per double, or 16 or 8 or 6 or 4 keys per double.
 - For each 6 keys in front row, 6 extra keys in back row (instead of 7 and 5)
- Every other or every third key in front row extends to back row.
- Examples of **Angel™**, **Elexys™**, and **Samira™** keyboards on next slides.

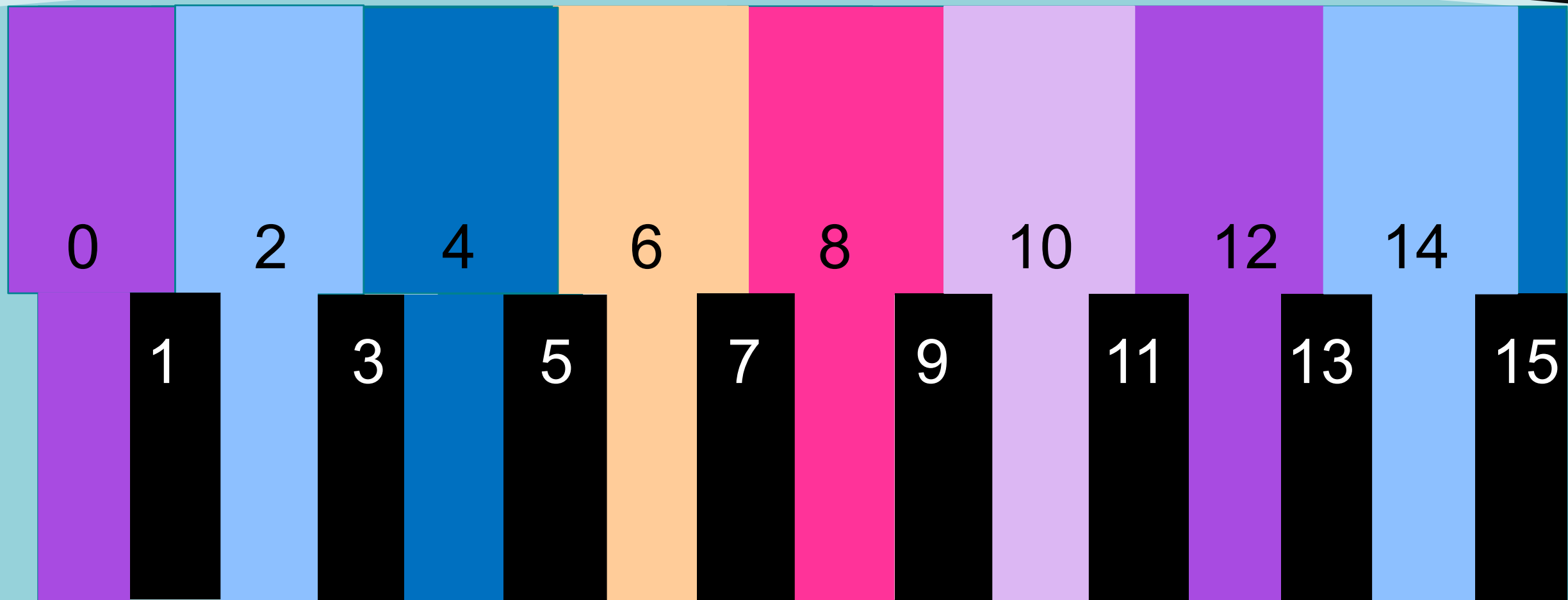
Angel™ piano keyboard with 3:2 ratio
Evenly spaced keys with ratio of 3 in front, 2 in back



Elexys™ piano keyboard with 4:3 ratio
Evenly spaced keys with 4 in front, 3 in back



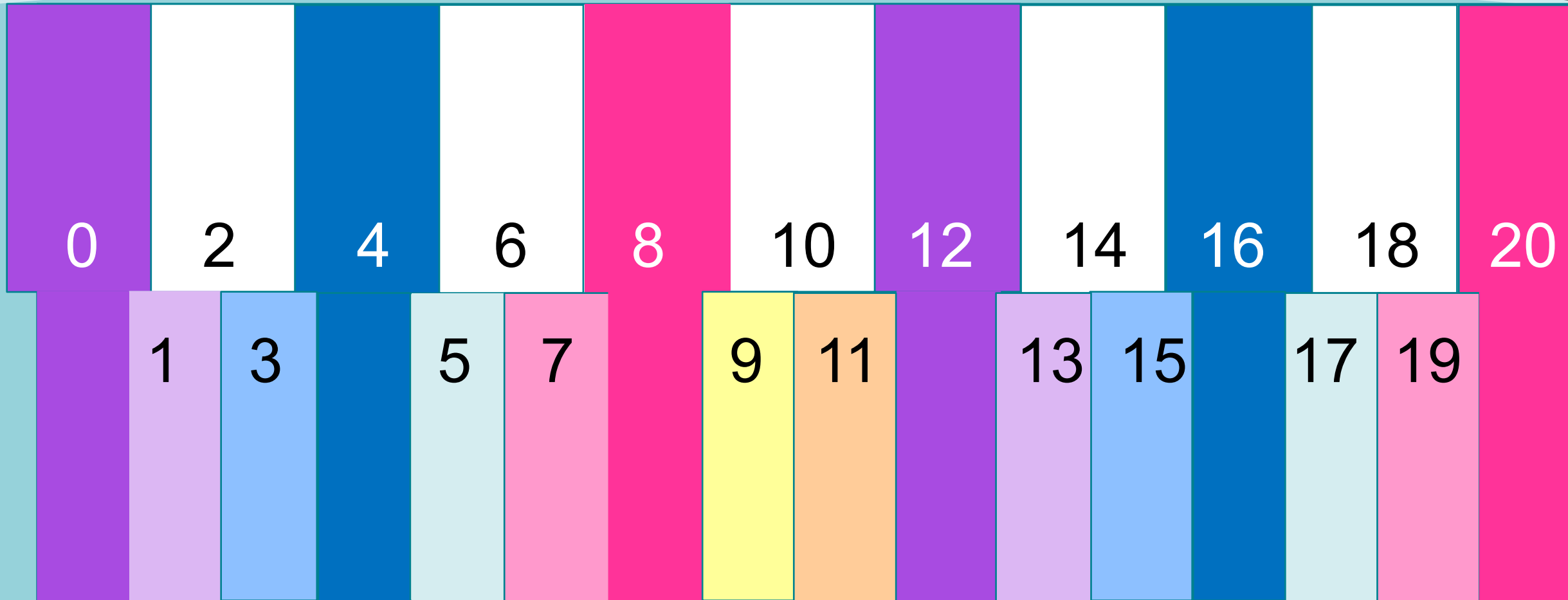
Samira™ piano keyboard with 2:1 ratio



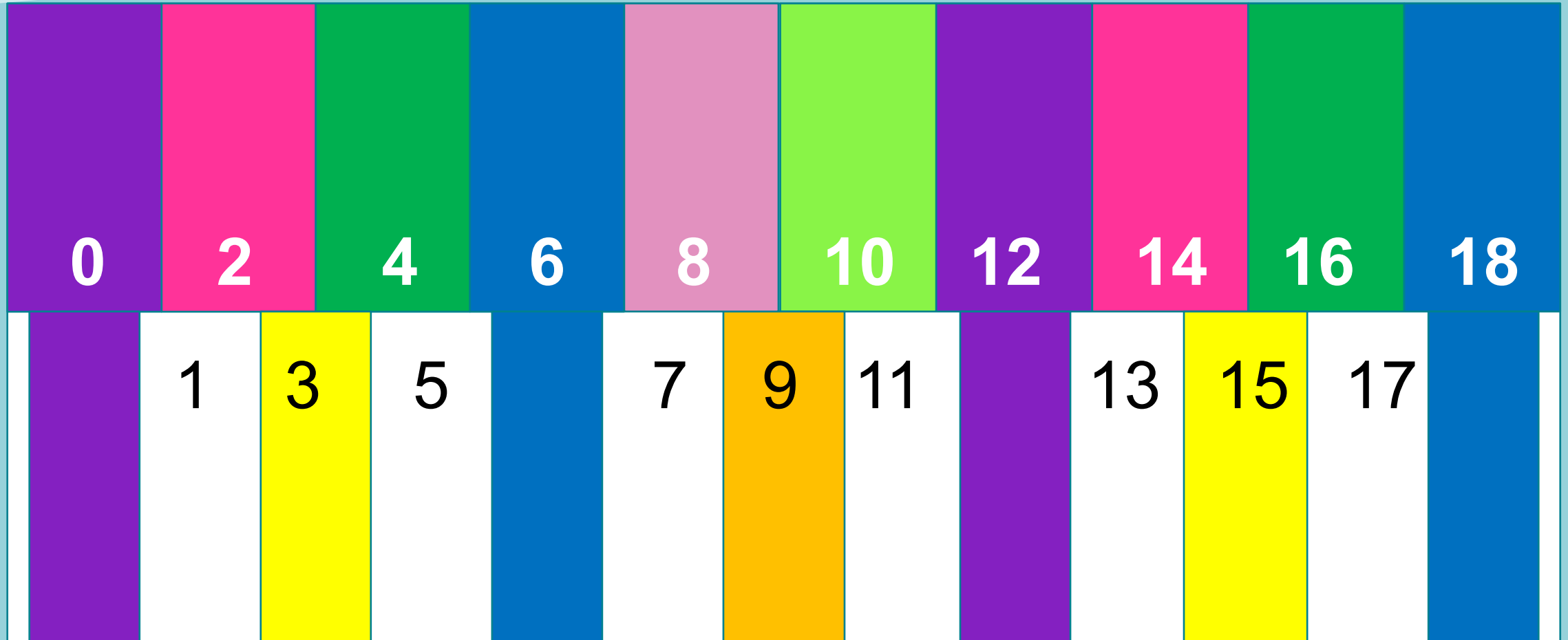
Coloring and key length options

- Color patterns are most useful if the tuning remains at 12 keys/double
 - For other widths the same colors may not match their frequency doubles, but then a row of lights could show which keys have double frequency.
 - Differing textures could also indicate every 4th, 6th, 8th, and 12th key
- Angel and Elexys keyboards have some tones only in the front row and other tones only in the back row. That can allow each hand to reach more different keys.
- Samira keyboard has even numbered keys (6 full steps) in one row and all keys (12 half steps) in the other row. The standard keyboard instead has an uneven pattern of 7 white + 5 black keys = 12 total.
- Beginners might use the row with fewer keys or reduce the keys/double, like using only the front row of the standard keyboard.

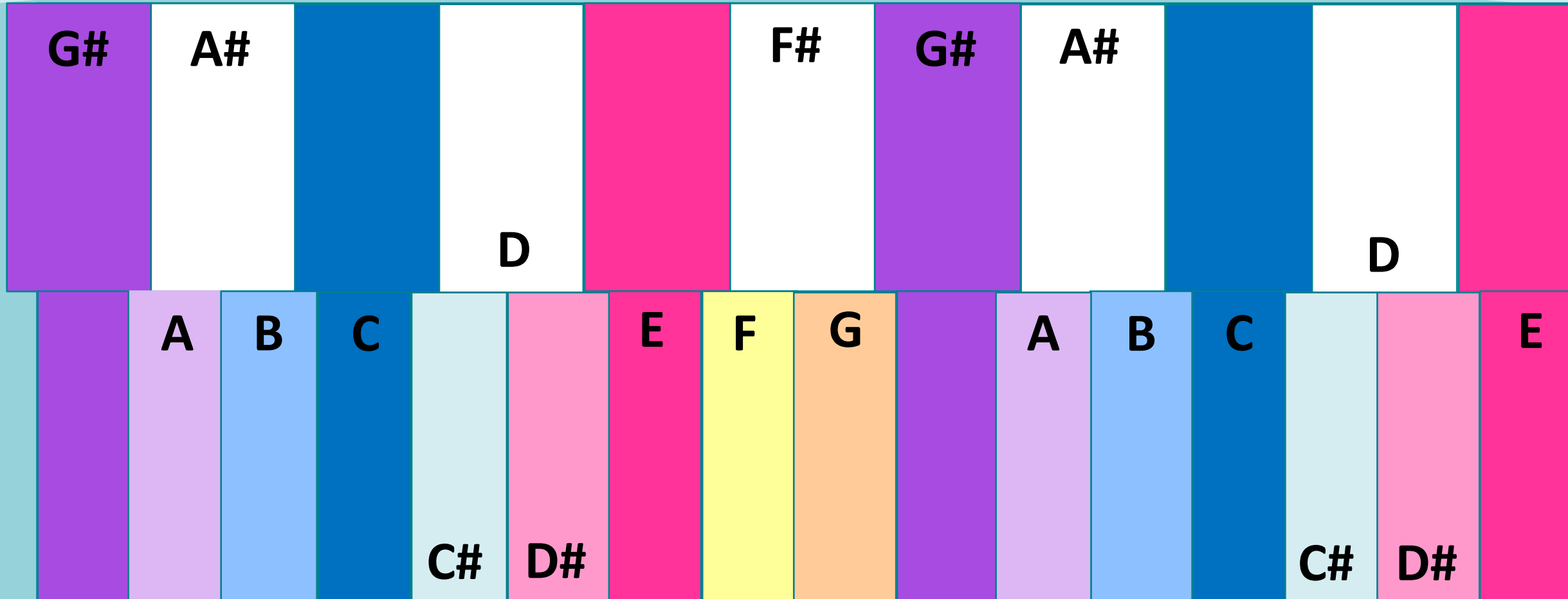
Key numbers on the *Angel*TM piano 3:2 keyboard



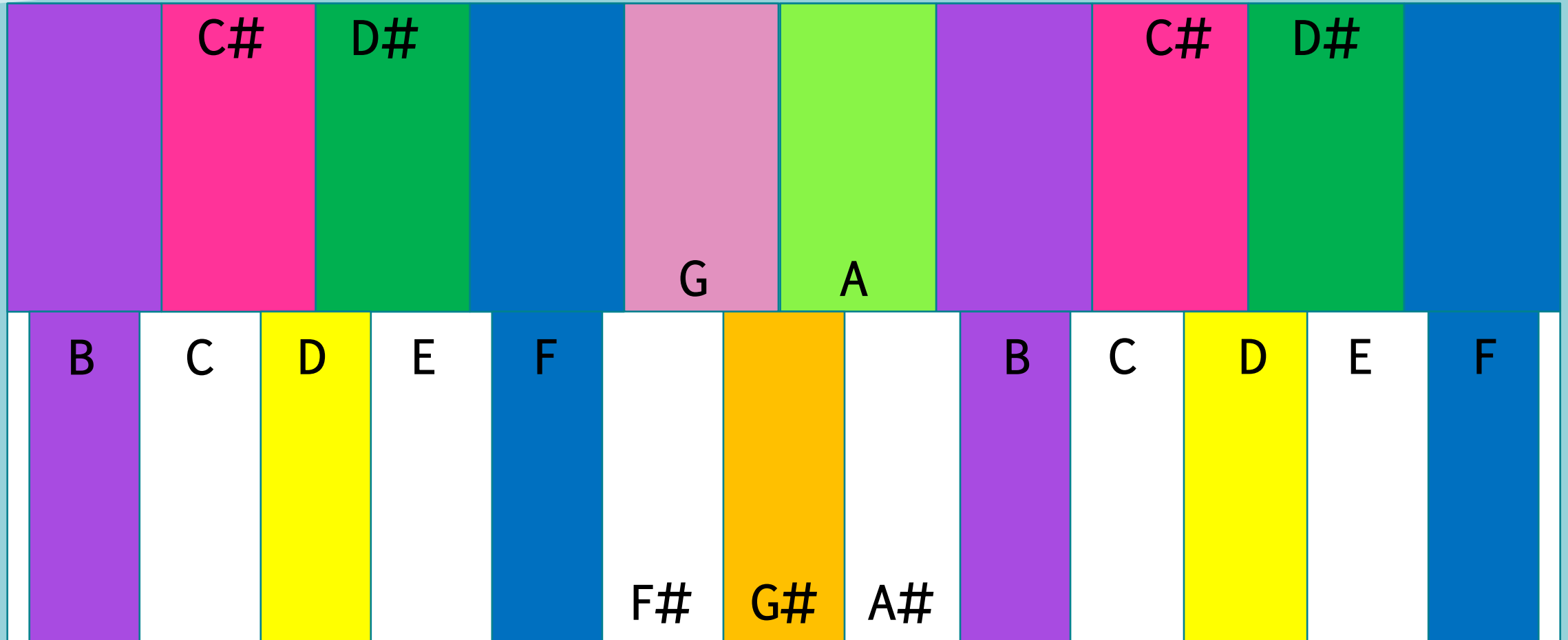
Key numbers on the Elexys™ piano 4:3 keyboard



Showing how to map standard keyboard notes
onto the *Angel*TM piano 3:2 keyboard



Showing how to map standard keyboard notes
onto the Elexys™ piano 4 : 3 keyboard



Keyboard tuning options and controls

- Width option sets number of notes per doubling of frequency
 - Preset selections on a knob or user input on a key scaling option screen
 - Default width: 12 notes per double
- Math option chooses from log, linear, half-linear, or reciprocal functions
 - Default math: log tuning
- Split option can tune the odd keys with a separate, smooth function
 - Interpolate between frequencies of the 2 nearest full-length keys
 - Increments = $1/3$ and $2/3$ for 3:2 keyboard or $1/4$, $1/2$, and $3/4$ for 4:3 keyboard
- Key numbers start with 0 at left using 1 sequence that includes both rows
 - Even numbers on the wider keys, odd numbers on the short, narrower keys
 - See next 2 slides

Example of width and split options

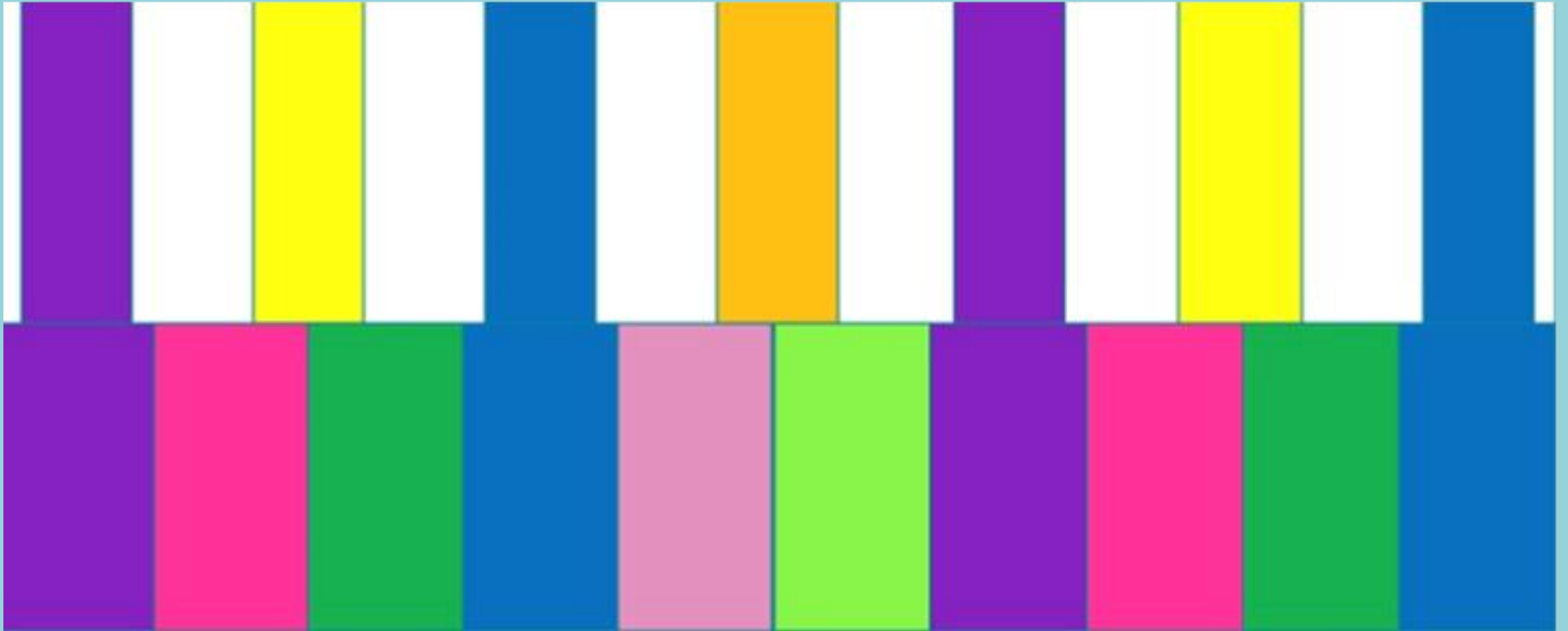
- Choose width = 8, math = linear, keyboard = 3:2
 - Back row key frequencies 1, 1.25, 1.5, 1.75, 2 with ratios 4:5:6:7:8
 - Front row keys 1, 1.125, 1.375, 1.5, 1.625, 1.875, 2 with ratios 8:9:11:12:13:15:16
 - Or split the odd keys in the front row into evenly spaced frequencies
 - Front row keys 1, 1.167, 1.333, 1.5, 1.667, 1.833, 2 with ratios 6:7:8:9:10:11:12
- Goals of tuning are
 - Keys next to each other should sound good when played together
 - Step sizes should be smooth with nice ratios instead of half and full steps
- Half-linear tuning options for splitting each double are on the next slide
 - Number of keys in each segment and frequency increases for those keys

Half-linear tuning choices for varying widths

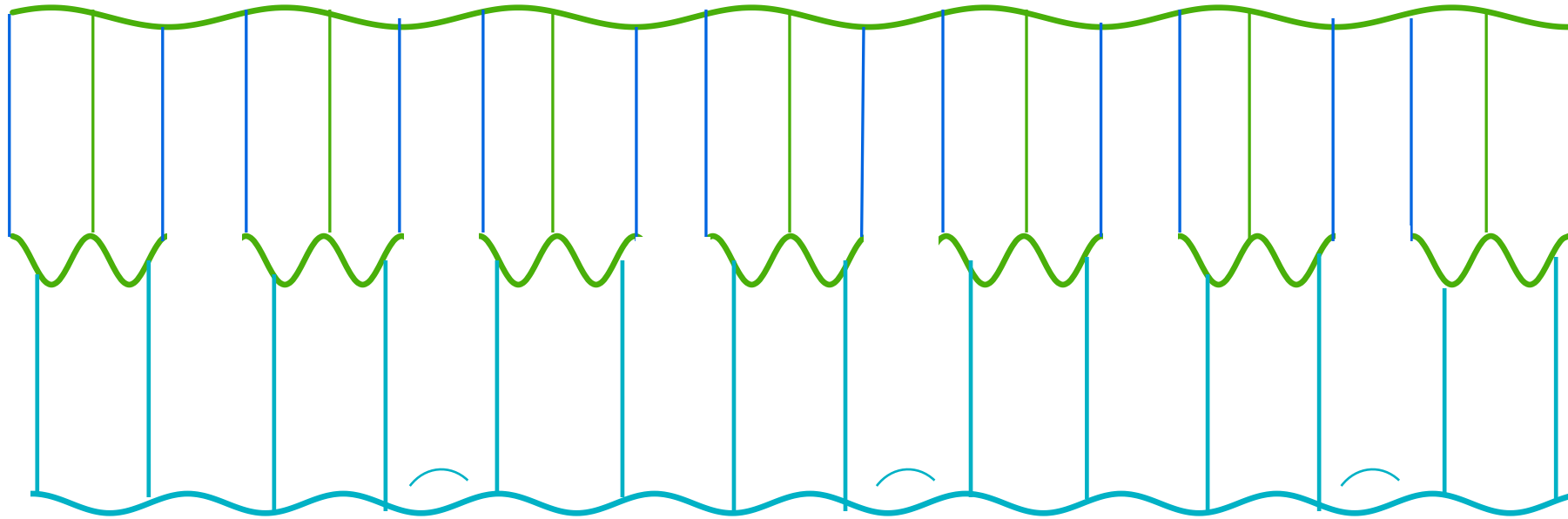
Width	Keys / half	Increments
5	3, 2	1/6, 1/4
6	4, 4	1/8, 1/4
7	3, 4	1/9, 1/6
8	5, 3	1/10, 1/6
9	5, 4	1/12, 1/8
10	6, 4	1/12, 1/8
11	9, 2	1/12, 1/8
12	6, 6	1/15, 1/10
13	9, 4	1/15, 1/10
14	6, 8	1/18, 1/12

Width	Keys / half	Increments
15	9, 6	1/6, 1/4
16	10, 6	1/20, 1/12
17	10, 7	1/20, 1/14
18	10, 8	1/20, 1/16
19	10, 9	1/20, 1/18
20	12, 8	1/24, 1/16
21	12, 9	1/24, 1/18
22	18, 4	1/24, 1/16
23	12, 11	1/24, 1/22
24	12, 12	1/30, 1/20

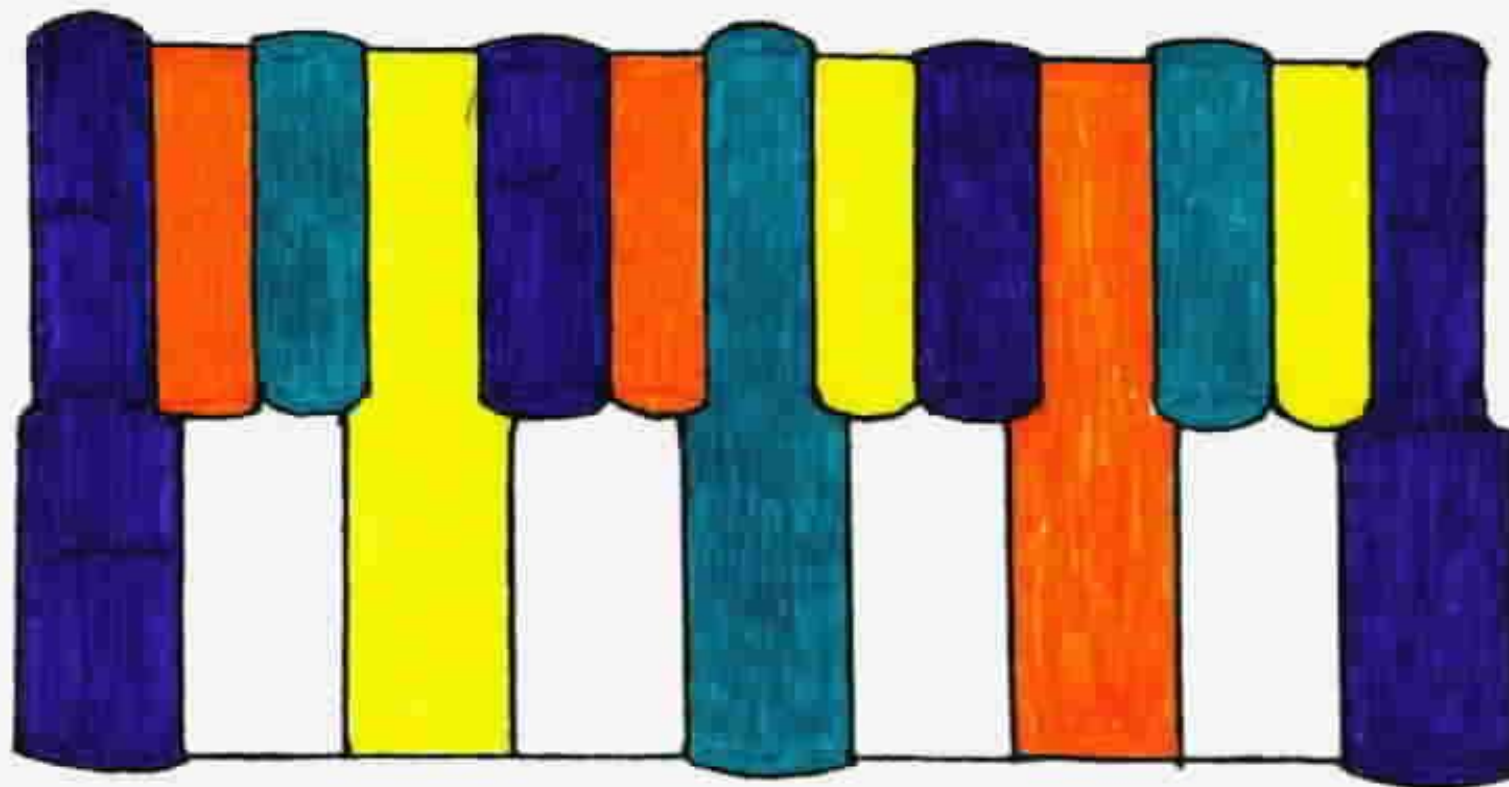
Evenly spaced keys with 3 in front, 4 in back
(upside down keyboard, 3:4 ratio)



Keyboard texture can help fingers find keys



Evenly spaced keys with 2 in front, 3 in back
Showing possible texture of keys (drawn in 1999)



Angel VanRaden wanted a new keyboard in 1999



Keyboard and tuning history

- Musical Instruments of the Ancient Greeks. 2009. [Konstantinos Kotsanas](#)
 - “The theory of musical intervals has suffered in modern European music the brutal blending for the benefit of polyphony at the expense of perfect natural musical intervals studied thoroughly by the ancient theorists of music (from Pythagoras and Aristogenes to Ptolemy).”
 - Pythagoras attached weights with ratios 4 : 6 : 8 : 9 : 12 : 16 to the strings of a musical instrument in about 500 BC.
 - Ctesibius used 12 tones per double and 24 pipes in 2 doubles in about 250 BC.
- Keyboard instrument - Special Arrangements | Britannica
- Keyboard instrument - Organ History, Baroque, Renaissance | Britannica
- Equal temperament log tuning using $2^{(n/12)}$ was derived by [Zhu Zaiyu, 1584](#)

Conclusions

- Ears detect harmony by repeating air pressure waves
- Linear tuning functions give more precise harmony than log tuning
- Much easier to put evenly spaced tones onto an evenly spaced keyboard
- Current 12 piano keys per double also fit onto an evenly spaced keyboard
- Electronic piano can be rescaled for beginners or professionals
- Several new tuning options are available
- Color coding and texture options make finding the keys easier
- Just need to build and sell a new keyboard to make huge profit

Evenly spaced keys with 4 in front, 3 in back,
slightly different color scheme

